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NON-LINEAR EFFECTS OF TRANSVERSE  
GRAVITY FIELD IN  
CAVITY FLOW

K.M. Agrawal

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June 1977

SCHOOL OF SCIENCE AND TECHNOLOGY  
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GRAVITY FIELD IN CAVITY FLOW.

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Technical report

by

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Krishan K.M. Agrawal

Sheau-Yen/Cheng

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### Abstract

A non-linear analysis has been made to determine the effects of the transverse gravity field on steady cavity flow past a symmetric wedge. Tulin's (1964) single-spiral cavity model is used to represent the flow and conformal mapping and Riemann-Hilbert technique have been employed to obtain the numerical solution. The present theory shows that the cavity length for the gravity solution is smaller and lies entirely within the gravity-free case. Also, the change in the drag coefficient from the non-gravity case to the gravity case is small.

### NOMENCLATURE

$A_0$  = the only non-zero power series coefficient

$A_j$  = constant, power series  $j$ th coefficient

$K^2$  = length of the cavity (from point of detachment to the rear of the cavity) ( $K^2 = C$ )

$H(t)$  = solution of the homogeneous Hilbert problem

$l$  = wedge arm length

$P_0$  = reference pressure at infinity

$p$  = pressure

$p_c$  = pressure in cavity

$q$  = magnitude of velocity

$q_0$  = magnitude of velocity at infinity

$Q(t)$  = solution of Riemann-Hilbert problem

$t$  = auxiliary half-plane

$t_1$  = real  $t$ -plane coordinate

$W$  = complex potential

$x$  = horizontal coordinate in the physical plane

$y$  = vertical coordinate in the physical plane

$\alpha$  = semi angle of wedge

$\zeta$  = normalized conjugate complex velocity

$\eta$  = dummy variable

$\rho$  = constant, density of fluid

$\sigma$  = cavitation number

$\tau$  = dummy variable

$\omega$  =  $\log \zeta$

$\phi$  = velocity potential

$\psi$  = stream function

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## 1. INTRODUCTION

Only a few theories have considered the influence of a gravity field. The actual magnitude of the effect of the gravity depends upon the Froude number and its orientation relative to the flow direction. In general, the Froude number attained in water tunnel tests indicate a small gravitational effects. However, the effect of the gravity may become important near free surface and certain other applications. The purpose of this work is to study the transverse gravity effect on a flow past a symmetric wedge, with a finite trailing cavity.

Parkin [1] and Street [2] have presented linearized theories for flows in a transverse gravity field using an additional approximation in the boundary conditions. Street [3] has this approximation to be quantitatively equivalent to a first-order solution in the small parameter  $1/F^2$ . In all these theories the assumption has been made that the ambient pressure is constant on each of the free streamlines, being lower than the reference pressure at the forebody on the upper, and higher than this reference pressure on the lower streamline.

The case of a longitudinal gravity field was studied by Acosta [4] and Lenau & Street [5]. Acosta [4] has developed a linearized theory by considering the flow past a symmetric wedge whereas Lenau & Street [5] have developed a non-linear theory by representing the cavitating flow



by a modified Riabouchinsky model. They have found that for a gravity field pointing the same direction as the flow the cavity is shortened, while for a gravity field in a direction pointing in a opposite to that of the flow, the cavity is lengthened. It is also interesting to note in these theories that after some critical values of the field strength, the trailing edge of the cavity becomes cusped rather than blunt. As mentioned by Tulin [6] real cavity flow involving longitudinal gravity fields are in most cases unsteady and the effect of the unsteadiness can easily predominate.

The effect of gravity on the stability of the cavity is discussed by Chang-Yi Wang [7].

## 2. GENERAL THEORY

We consider the steady, two-dimensioned irrotational flow of an inviscid, incompressible fluid past a fully cavitating symmetric wedge of angle  $2\alpha$  under the influence of transverse gravity. By fully cavitating flow we mean a cavity extending along the entire suction side and even beyond the trailing edge. Bernoulli's theorem can be written

$$p + 1/2 \rho q^2 + g \rho y = \text{constant} \quad (1)$$

where  $p$  - is the pressure

$q$  - is the speed of the fluid

$y$  - is the vertical distance between a point in the fluid and some reference elevation

$\rho$  - is the fluid density

$g$  - is the acceleration due to gravity



We define the cavitation number

$$\sigma = \frac{P_0 - P_c}{\frac{1}{2} \rho q_0^2} \quad (2)$$

and the Froude number

$$F = \sqrt{\frac{q_0^2}{gl}} \quad (3)$$

where  $l$  is some characteristic length, and will be taken as the wedge length in what follows. In the presence of gravity, the fluid speed  $q$  can be expressed in terms of  $q_0$  and the Froude number by

$$\frac{q}{q_0} = \left(1 - \frac{1}{1+\sigma} \cdot \frac{1}{F^2} \cdot \frac{2l}{l}\right)^{1/2} \quad (4)$$

We also define the following symbols :

$$Z = x + iy$$

$$W = \Phi + i\Psi$$

$$\frac{dW}{dz} = \zeta = u - iv = q e^{-i\theta} \quad (5)$$

$$\omega = \log \frac{dW}{dz} = \log q + i(-\theta) \quad (6)$$

The method of solution consists of two basic parts :

- (a) Construction of conformal mapping between the physical plane and the complex potential plane and a parametric half plane.
- (b) Solution in this plane of a well posed Riemann-Hilbert problem.

Towards this goal we can determine  $W$  as a function of the half-plane variable  $t$  by conformal mapping. Then by using the Riemann-Hilbert technique we construct  $\omega(t)$  explicitly as a function of  $t$  and once  $\omega(t)$

is known, the drag coefficient can be determined using Bernoulli's theorem.

The general solution of the Riemann-Hilbert problem in the upper-half plane is well known [8,9]. In this problem if the imaginary part of an analytic function  $Q(t)$  is known at all points on the boundary, i.e., on the real  $t$ -axis, then  $Q(t)$  is obtained in the entire upper-half plane from the formula :

$$Q(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im}[Q(z)]}{z-t} dz + \sum_{j=0}^{\infty} A_j t^j \text{-----} (7)$$

where  $A_j$  are real constants and  $j$  represents all positive integers.

Now, it remains to relate  $\omega(t)$  to  $Q(t)$  so that  $\text{Im } Q(t)$  is known at every point on the real line.

We map  $W$ -plane onto the  $t$ -plane so that the wedge is mapped onto a finite segment of the real axis, and the cavity boundary onto the remainder of the real axis. Then, we have boundary conditions along the real axis in terms of  $\omega(t)$ .

We can convert these conditions to those of the Riemann-Hilbert problem following the general method of solution reported by Cheng and Rott [10]. The original function  $\omega(t)$  is alternately purely real or purely imaginary on the real axis, hence we construct an auxiliary function  $H(t)$  which makes the quotient  $Q(t) = \frac{\omega(t)}{H(t)}$  known on the entire real axis, and always imaginary as required. It is to be noted that  $H(t)$  is a solution of the homogeneous mixed boundary value problem; i.e., the prescribed values of  $\omega(t)$  are all zero along the real axis. The choice of the homogeneous solution  $H(t)$  is quite arbitrary.

A choice of  $H(t)$  satisfying our requirement is ( see Fig. 3 ):

$$H(t) = 1 \left[ (t-t_1)(t-t_2) \right]^{1/2}, \quad t_1 < t_2 \text{ --- (8)}$$

where  $t_1$  and  $t_2$  correspond to the ends of the wedge.

Let 
$$\text{Im} [H(t)] > 0$$

on the real axis for  $t < t_1$ , and select a branch cut on the real axis interval  $(t_1, t_2)$  so that  $H(t)$  is single valued. Hence  $\omega(t)$  can explicitly be constructed by the use of the Riemann-Hilbert solution where  $\omega(t) = H(t) Q(t)$ .

### 3. ANALYSIS

The solution to our problem is based on an iterative procedure suggested by Larock and Street [11] which employs the non-gravity solution as the initial trial solution (The non-gravity solution is obtained from Ref. 12). We are using the same cavity model (single spiral vortex model) as is used in the non-gravity case. There are two distinct numerical advantages in using this particular model as discussed in Ref. 11 (see also figures 1, 2 and 3).

Let us denote  $y_L$  and  $y_U$  be the respective values of  $y(t)$  for the lower and upper streamlines beyond which  $y(t)$  becomes constant (Fig. 1) and corresponding values of  $t$  are  $t_L$  and  $t_U$ . Then, for  $t \leq t_L$ ,  $y(t) = y_L$ ; for  $t \geq t_U$ ,  $y(t) = y_U$ .

In this problem the gravity acts parallel to the  $y$ -axis and in the negative direction. The formulation can now be completed by starting the boundary conditions. They are:

$$\begin{aligned} 1. \quad q &= q_c \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2\gamma}{\lambda} \right)^{1/2}, & \text{on the cavity and as } z \rightarrow \infty \\ \log \frac{q}{q_c} &\rightarrow -\frac{1}{2} \log(1+\sigma), & \text{Im}(\omega(t)) = 0, \quad t = ik \end{aligned}$$

2. The combination of the body and cavity must be closed. The equivalent statement is that the net source strength must be zero [12].

3. Lastly, the flow cannot contain non-integrable singularities on the slit or have multiple values off the slit [13].

Conditions 1 - 3 are sufficient to determine the flow field.

Using the Ref. 12 we can construct the following boundary value problem :

$$\begin{aligned}
 \operatorname{Re}[\omega] &= \frac{1}{2} \log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y(t)}{l} \right], & -\infty < t < -1 \\
 \operatorname{Im}[\omega] &= \alpha, & -1 < t < 0 \\
 \operatorname{Im}[\omega] &= -\alpha, & 0 < t < 1 \\
 \operatorname{Re}[\omega] &= \frac{1}{2} \log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y(t)}{l} \right], & 1 < t < \infty
 \end{aligned} \tag{9}$$

where  $F$  is a prescribed Froude number;  $\alpha$ ,  $\sigma$ ,  $l$  and  $y(t)$  are respectively, the semi-wedge angle, cavitation number, wedge length and the ordinate of the free streamline for the gravity problem.

In order to apply the Riemann-Hilbert technique, we now form  $\frac{\omega(t)}{H(t)}$  [12] as follows :

$$\begin{aligned}
 \operatorname{Im} \left[ \frac{\omega(t)}{H(t)} \right] &= -\frac{1}{2} \log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y(t)}{l} \right] [(1+t)(t-1)]^{-1/2}, & -\infty < t < -1 \\
 \operatorname{Im} \left[ \frac{\omega(t)}{H(t)} \right] &= \alpha [(1-t)(1+t)]^{-1/2}, & -1 < t < 0 \\
 \operatorname{Im} \left[ \frac{\omega(t)}{H(t)} \right] &= -\alpha [(1-t)(1+t)]^{-1/2}, & 0 < t < 1 \\
 \operatorname{Im} \left[ \frac{\omega(t)}{H(t)} \right] &= \frac{1}{2} \log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y(t)}{l} \right] [(1+t)(t-1)]^{-1/2}, & 1 < t < \infty
 \end{aligned} \tag{10}$$



The Riemann-Hilbert power series will still give only one term as in Ref. 12 . Hence

$$\begin{aligned} \omega(t) = H(t) \left\{ -\frac{1}{2\pi} \int_{-\infty}^{-1} \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{z^2(\eta)}{L} \right]}{(\eta-t)(\eta^2-1)^{1/2}} d\eta \right. \\ + \frac{\alpha}{\pi} \int_{-1}^0 \frac{d\eta}{(\eta-t)(1-\eta^2)^{1/2}} - \frac{\alpha}{\pi} \int_0^1 \frac{d\eta}{(\eta-t)(1-\eta^2)^{1/2}} \\ \left. + \frac{1}{2\pi} \int_1^{\infty} \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{z^2(\eta)}{L} \right]}{(\eta-t)(\eta^2-1)^{1/2}} d\eta + A_0 \right\} \quad (11) \end{aligned}$$

Here, two terms have been added to the non-gravity expression for  $\omega(t)$  i.e., the integrals over  $(-\infty, -1)$  and  $(1, \infty)$ .

In order to evaluate the extra integrals we will follow the Larock and Street method [11]. Recalling that  $y(t) = y_L$ , for  $t \leq t_L$ ; and  $y(t) = y_U$ , for  $t \geq t_U$ . We obtain

$$\begin{aligned} \int_{-\infty}^{-1} \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{z^2(\eta)}{L} \right]}{(\eta-t)(\eta^2-1)^{1/2}} d\eta = \log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{z^2(\eta)}{L} \right) \\ + \int_{-\infty}^{t_L} \frac{d\eta}{(\eta-t)(\eta^2-1)^{1/2}} + \int_{t_L}^{-1} \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{z^2(\eta)}{L} \right]}{(\eta-t)(\eta^2-1)^{1/2}} d\eta \end{aligned}$$

Likewise,

$$\begin{aligned} \int_1^{\infty} \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{z^2(\eta)}{L} \right]}{(\eta-t)(\eta^2-1)^{1/2}} d\eta = \log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{z^2(\eta)}{L} \right) \\ + \int_{t_U}^{\infty} \frac{d\eta}{(\eta-t)(\eta^2-1)^{1/2}} + \int_1^{t_U} \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{z^2(\eta)}{L} \right]}{(\eta-t)(\eta^2-1)^{1/2}} d\eta \end{aligned}$$

The integrals over  $(-\infty, t_L)$  and  $(t_U, \infty)$  can be evaluated exactly.

The unknowns of the problem are the same as in the non-gravity case. We can rewrite Eq. ( 11 ) as



$$\begin{aligned}
\omega(t) = H(t) & \left\{ -\frac{1}{2\pi} \log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g_L}{L}\right) \int_{-\infty}^{t_U} \frac{d\eta}{(\eta-t)(\eta^2-1)^{1/2}} \right. \\
& + \frac{\alpha}{\pi} \int_{-1}^0 \frac{d\eta}{(\eta-t)(1-\eta^2)^{1/2}} - \frac{\alpha}{\pi} \int_0^1 \frac{d\eta}{(\eta-t)(1-\eta^2)^{1/2}} \\
& + \frac{1}{2\pi} \log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g_U}{L}\right) \int_{t_U}^{\infty} \frac{d\eta}{(\eta-t)(\eta^2-1)^{1/2}} - \frac{1}{2\pi} \int_{t_U}^{-1} \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{L}\right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} \\
& \left. + \frac{1}{2\pi} \int_1^{t_U} \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{L}\right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} + A_0 \right\} \quad \text{----- (12)}
\end{aligned}$$

On the real  $t$ -axis, taking the proper branch of  $H(t)$  in the different region, we obtain [12] :

$$\begin{aligned}
\omega(t) = & -\frac{\alpha}{\pi} \left[ -i\pi - \log\left\{ \frac{1-(1-t^2)^{1/2}}{1+(1-t^2)^{1/2}} \right\} \right] + A_0 (1-t^2)^{1/2} + J_L(t) \\
& + \frac{i \log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g_L}{L}\right)}{2\pi (1-t^2)^{1/2}} \left[ \log\left\{ \frac{(1+t)^{1/2}(1-t_L)^{1/2} - (1-t)^{1/2}(1+t_L)^{1/2}}{(1+t)^{1/2}(1-t_L)^{1/2} + (1-t)^{1/2}(1+t_L)^{1/2}} \right\} \right. \\
& \quad \left. - \log\left\{ t - i(1-t^2)^{1/2} \right\} \right] \\
& + \frac{i \log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g_U}{L}\right)}{2\pi (1-t^2)^{1/2}} \left[ \log\left\{ t - i(1-t^2)^{1/2} \right\} \right. \\
& \quad \left. - \log\left\{ \frac{(1+t)^{1/2}(1-t_U)^{1/2} - (1-t)^{1/2}(1+t_U)^{1/2}}{(1+t)^{1/2}(1-t_U)^{1/2} + (1-t)^{1/2}(1+t_U)^{1/2}} \right\} \right] \\
& \quad \quad \quad -1 \leq t \leq 0 \\
& \quad \quad \quad \text{----- (13)}
\end{aligned}$$

$$\begin{aligned}
\omega(t) = & -\frac{\alpha}{\pi} \left[ -i\pi - \log \left\{ \frac{1-(1-t^2)^{1/2}}{1+(1-t^2)^{1/2}} \right\} \right] + A_0(1-t^2)^{1/2} + J_L(t) \\
& + \frac{i \log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y_L}{L} \right)}{2\pi(1-t^2)^{1/2}} \left[ \log \left\{ \frac{(1+t)^{1/2}(1-t_L)^{1/2} - (1-t)^{1/2}(1+t_L)^{1/2}}{(1+t)^{1/2}(1-t_L)^{1/2} + (1-t)^{1/2}(1+t_L)^{1/2}} \right\} \right. \\
& \quad \left. - \log \left\{ t - i(1-t^2)^{1/2} \right\} \right] \\
& + \frac{i \log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y_U}{L} \right)}{2\pi(1-t^2)^{1/2}} \left[ \log \left\{ t - i(1-t^2)^{1/2} \right\} \right. \\
& \quad \left. - \log \left\{ \frac{(1+t)^{1/2}(1-t_U)^{1/2} - (1-t)^{1/2}(1+t_U)^{1/2}}{(1+t)^{1/2}(1-t_U)^{1/2} + (1-t)^{1/2}(1+t_U)^{1/2}} \right\} \right]
\end{aligned}$$

where

$$0 \leq t \leq 1$$

----- (14)

$$J_L(t) = \frac{(1-t^2)^{1/2}}{2\pi} \left[ \int_1^{t_U} \frac{\log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y(\eta)}{L} \right)}{(\eta-t)(\eta^2-1)^{1/2}} d\eta - \int_{t_L}^{-1} \frac{\log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y(\eta)}{L} \right)}{(\eta-t)(\eta^2-1)^{1/2}} d\eta \right]$$

----- (15)

$$\begin{aligned}
\omega(t) = & -\frac{\alpha}{\pi} \left[ \mp i\pi + 2i \arctan(t^2-1)^{1/2} \right] \pm i A_0(t^2-1)^{1/2} \pm i J(t) \\
& + \frac{\log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y_L}{L} \right)}{2\pi(t^2-1)^{1/2}} \left[ \log \left\{ \frac{(1+t)^{1/2}(1-t_L)^{1/2} - (1-t)^{1/2}(1+t_L)^{1/2}}{(1+t)^{1/2}(1-t_L)^{1/2} + (1-t)^{1/2}(1+t_L)^{1/2}} \right\} \right. \\
& \quad \left. - \log \left\{ t \pm (t^2-1)^{1/2} \right\} \right] \\
& + \frac{\log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y_U}{L} \right)}{2\pi(t^2-1)^{1/2}} \left[ \log \left\{ \frac{(1+t)^{1/2}(1-t_U)^{1/2} - (1-t)^{1/2}(1+t_U)^{1/2}}{(1+t)^{1/2}(1-t_U)^{1/2} + (1-t)^{1/2}(1+t_U)^{1/2}} \right\} \right. \\
& \quad \left. + \log \left\{ t \pm (t^2-1)^{1/2} \right\} \right]
\end{aligned}$$

For  $t \leq -1$ , take (+) sign

For  $t \geq 1$ , take (-) sign

----- (16)

where

$$J(t) = \frac{(t^2-1)^{1/2}}{2\pi} \left[ \int_1^{t-\varepsilon} \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{\eta}\right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} + \int_{t+\varepsilon}^t \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{\eta}\right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} \right. \\ \left. - \int_{t_L}^{-1} \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{\eta}\right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} \right]$$

For  $t > 1$

----- (17)

$$J(t) = \frac{(t^2-1)^{1/2}}{2\pi} \left[ \int_1^t \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{\eta}\right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} - \int_{t_L}^{t-\varepsilon} \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{\eta}\right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} \right. \\ \left. - \int_{t+\varepsilon}^{-1} \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{\eta}\right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} \right]$$

For  $t < -1$

----- (18)

As in the non-gravity case, we have two unknowns  $A_0$  and  $K$ . Since boundary conditions at infinity ( $t=ik$ ) still hold, the following integrals (using the notation of Larock and Street [11]) will be found useful :

$$I_1(\eta_1, \eta_2) = \int_{\eta_1}^{\eta_2} \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{\eta}\right) d\eta}{(\eta^2+K^2)(\eta^2-1)^{1/2}}$$

$$I_2(\eta_1, \eta_2) = \int_{\eta_1}^{\eta_2} \frac{\log\left(1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{\eta}\right) d\eta}{(\eta^2+K^2)^2(\eta^2-1)^{1/2}}$$

A numerical investigation shows that the integrals (Eq. 12 ) over the interval  $(t_U, \infty)$  and  $(-\infty, t_L)$  ( these integrals represent the spiralling region of free streamlines ) have very little effect on the other integrals. Therefore, in further calculation we will neglected these integrals.

The two equations for determining the unknowns (similar to non-gravity case, Ref. 12 ) are :

$$-\frac{1}{2} \log(1+\sigma) = \frac{\alpha}{\pi} \log \left\{ \frac{(1+k^2)^{1/2} - 1}{(1+k^2)^{1/2} + 1} \right\} + \frac{(1+k^2)^{1/2}}{2\pi} \cdot \left\{ I_1(1, t_U) - I_1(t_L, -1) \right\} + A_0 (1+k^2)^{1/2} \quad (19)$$

$$\frac{-2\alpha}{\pi k(k^2+1)^{1/2}} + \frac{1}{2\pi} \left[ -\frac{k}{(1+k^2)^{1/2}} \left\{ I_1(1, t_U) - I_1(t_L, -1) \right\} + 2k(1+k^2)^{1/2} \left\{ I_2(1, t_U) - I_2(t_L, -1) \right\} \right] - \frac{KA_0}{(1+k^2)^{1/2}} = 0 \quad (20)$$

From these two equations we can always find  $A_0$ ,  $K$  for a given  $\alpha$ ,  $\sigma$ ,  $t_L$  and  $t_U$ .

#### 4. CALCULATIONS FOR WEDGE LENGTH, DRAG AND LIFT

The expression for the wedge arm length is :

$$l = \int_A^B |dz| = \int_0^1 2k^2(k^2+1) e^{-A_0(1-t^2)^{1/2} - J_L(t)} \left[ \frac{1+(1-t^2)^{1/2}}{1-(1-t^2)^{1/2}} \right]^{\alpha/\pi} \frac{t dt}{(t^2+k^2)^2}$$

The expression for the lift and drag is given by :

$$D + iL = -i \int_A^B (P - P_c) dz + i \int_A^{\bar{B}} (P - P_c) d\bar{z}$$



From the Bernoulli's equation we have

$$\frac{P - P_c}{\frac{1}{2} \rho g_c^2} = \left[ 1 - \left( \frac{g}{g_c} \right)^2 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y}{l} \right]$$

Defining the lift and drag coefficients, respectively, as

$$C_L = \frac{L}{\frac{1}{2} \rho g_c^2 l}, \quad C_D = \frac{D}{\frac{1}{2} \rho g_c^2 l}$$

We have

$$\begin{aligned} C_D + i C_L &= -\frac{i}{l} \int_A^B \left[ 1 - \left( \frac{g}{g_c} \right)^2 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y}{l} \right] dz \\ &\quad + \frac{i}{l} \int_A^{\bar{B}} \left[ 1 - \left( \frac{g}{g_c} \right)^2 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2y}{l} \right] dz \\ &= 2 \sin \alpha \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \sin \alpha \right) + \frac{i}{l} \int_A^B \left( \frac{g}{g_c} \right)^2 dz - \frac{i}{l} \int_A^{\bar{B}} \left( \frac{g}{g_c} \right)^2 dz \\ &= 2 \sin \alpha \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \sin \alpha \right) + \frac{i}{l} \int_0^1 e^{\bar{\omega}} \frac{d\omega}{dt} - \frac{i}{l} \int_0^{-1} e^{\bar{\omega}} \frac{d\omega}{dt} \\ &= 2 \sin \alpha \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \sin \alpha \right) + \frac{i}{l} \left\{ 2 \int_0^1 e^{-i\alpha + A_0(1-t^2)^{1/2} + J_R(t)} \right. \\ &\quad \cdot \left[ \frac{1 - (1-t^2)^{1/2}}{1 + (1-t^2)^{1/2}} \right]^{\alpha/\pi} \cdot \frac{t k^2 (k^2 + 1) dt}{(t^2 + k^2)^2} \\ &\quad \left. + 2 \int_{-1}^0 e^{-i\alpha + A_0(1-t^2)^{1/2} + J_R(t)} \cdot \left[ \frac{1 - (1-t^2)^{1/2}}{1 + (1-t^2)^{1/2}} \right]^{\alpha/\pi} \cdot \frac{t k^2 (k^2 + 1) dt}{(t^2 + k^2)^2} \right\} \end{aligned}$$



Let

$$I_C = \frac{2k^2(k^2+1)}{2} \int_{t_1}^t e^{A_0(1-t^2)^{1/2} + J_2(t)} \left[ \frac{1-(1-t^2)^{1/2}}{1+(1-t^2)^{1/2}} \right]^{1/2} \frac{t dt}{(t^2+k^2)^2}$$

Then,

$$C_D = 2 \sin \alpha \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \sin \alpha \right) + (I_{C_2} - I_{C_1}) \sin \alpha \quad (21)$$

$$C_L = (I_{C_1} + I_{C_2}) \cos \alpha \quad \text{--- (22)}$$

where  $I_{C_1}$  is the value of  $I_C$  with limits 0 to 1; and  $I_{C_2}$  is the value of  $I_C$  with limits -1 to 0.

The parametric representation of the free streamline is given by

$$Z - Z_0 = \int_{t_0}^t e^{-\omega(t)} \frac{dw}{dt} dt \quad \text{--- (23)}$$

where  $\omega(t)$  is given by Eq. (14) and  $t_0 = 1$  for the upper streamline and  $t_0 = -1$  for the lower streamline.

## 5. NUMERICAL PROCEDURE

The iterative method which we have used in finding the numerical solution to the present problem is described in detail by Larock and Street [11]. The iterative procedure works in the following manner :

- (1) We determine all the parameters and the physical plane, especially  $y(t)$  on the streamlines, from the non-gravity case (Ref. 12) for a given  $K$  and  $\alpha$ .
- (2) Using the input data from step (1) where appropriate  $I_1$ ,  $I_2$  and  $A_0$  (Eqs. 19, 20) are then computed. Then we determine  $J(t)$ ,  $y(t)$  and  $J_2(t)$  (Eqs. 18, 23 & 15)
- (3) Another gravity solution is obtained using the data from the step (2) as input data.

(4) Step (3) is repeated until the results of two successive integrations differ by less than some specified amount; at this point we stop the calculations and gravity solution is obtained.

The method just described is direct and simple to apply. However, there is one drawback to this method, namely, that it is very time-consuming. Even computation of one iteration takes a considerable time on a high speed computer.

First,  $y(t)$  or some approximation to  $y(t)$  is assumed to be known as a continuous function of  $t$ . The function  $J(t)$  and  $J_g(t)$  must also be determined as continuous function of  $t$  since they occur in various integrands. It is not possible to compute the value of these functions for all  $t$  because it would be a very tedious and expensive process. Therefore,  $y(t)$ ,  $J(t)$  and  $J_g(t)$  are computed for a selected, finite number of points  $t_1$ . Then, by using interpolation methods these functions are approximated between the points  $t_1$ .

In numerical calculations, proper care must be taken at the singularities of the integrands, even though a singularity is integrable.

In Eq. 18, valid for  $t < -1$ , the integrand over the interval  $(1, t_U)$  has a square root singularity at 1. We treat the remainder of the integrand as a constant over the interval  $(1, 1+\epsilon)$  and integrate the singular portion:

$$\int_1^{t_U} \frac{\log \left( 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g(\eta)}{2} \right) d\eta}{(\eta-t)(\eta^2-1)^{1/2}} = \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2g_0}{2} \right]}{2(1-t)} \int_1^{1+\epsilon} \frac{d\eta}{(\eta-1)^{1/2}}$$

$$\begin{aligned}
& + \int_{1+\epsilon}^{t_U} \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2\gamma(\eta)}{\eta} \right] d\eta}{(\eta-t)(\eta^2-1)^{1/2}} \\
& = \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2\gamma_0}{\eta} \right]}{2(1-t)} \cdot 2\epsilon^{1/2} + \int_{1+\epsilon}^{t_U} \frac{\log \left[ 1 - \frac{1}{1+\sigma} \frac{1}{F^2} \frac{2\gamma(\eta)}{\eta} \right] d\eta}{(\eta-t)(\eta^2-1)^{1/2}}
\end{aligned}$$

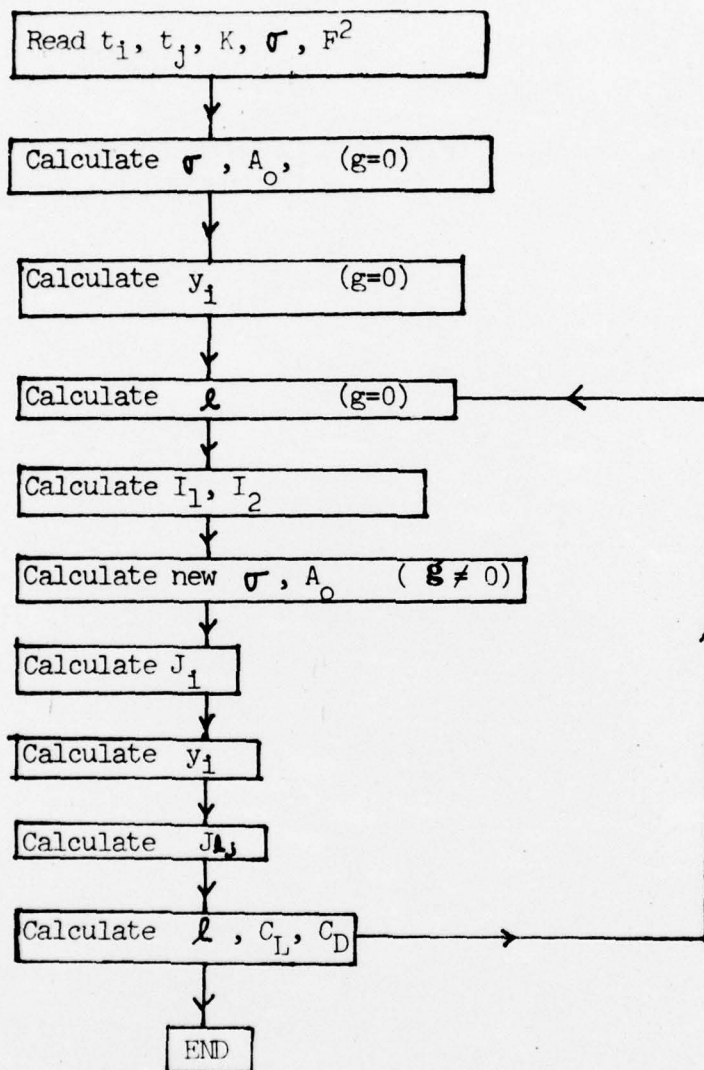
For a specified  $t$  and  $\epsilon$ , the term can now be computed numerically.

So long as  $0 < \epsilon < 1$ , the specific choice of  $\epsilon$  is not important.

Likewise, we can treat the integral over the interval  $(t_L, -1)$ .

As pointed out in Ref. 12, the points where the streamlines are vertical at the end of the cavity can be found as a function of  $t$  by finding the first zero of the integrand of equation (4.14) in Ref. 12. We call this point  $t_U$ . Since we are considering a symmetric wedge flow in the non-gravity case, the value of  $t_L$  will be the negative of  $t_U$ . (The roots which make the integrand zero are obtained by a bi-section procedure.)

## FLOW CHART





## 6. DISCUSSION

Figures 4 and 5 show the effect of the Froude number on the cavity shape. The cavity for the gravity solution is smaller and lies entirely within the gravity-free case. Also, the change in the drag coefficient from the non-gravity case to the gravity case is small.

From the present theory there results a negative lift coefficient. Intuitively, it seems that it will become positive only when the cavity is so short that its downward pulling effect is balanced by the buoyant force of the free body.

As discussed by Larock and Street [11] and Parkin [1] it can be shown in the present case also that the effect of the gravity increases with decreasing cavitation number. ( This was not done because of computer time constraint .) This simply means that longer cavities are more affected by the gravity than the short ones.



## References

1. Parkin, B.R., 'A Note on the Cavity Flow Past a Hydrofoil in a Liquid with Gravity', Report No. 47-9 Engineering Division, California Tech. Pasadena, 1957.
2. Street, R.L., 'Supercavitating Flow about a Slender Wedge in a Transverse Gravity Field', Journal of Ship Research, 7,1, pp. 14-23, 1963.
3. Street, R.L., 'A note on Gravity Effects in Supercavitating Flow', Journal of Ship Research, 8,1, pp. 39-45, 1965
4. Acosta, A.J., 'The Effects of a Longitudinal Gravitational Field on the Supercavitating Flow over a Wedge', Journal of Applied Mechanics, 28, Trans. ASME, 83, Series E, pp. 188-192, 1961
5. Lenau, C.W. and Street, R.L., 'A Non-linear Theory for Symmetric Supercavitating Flow in a Gravity Field', Journal of Fluid Mechanics, 21, 2, pp. 257-280, 1965.
6. Tulin, M.P., 'The Shape of Cavities in Supercavitating Flows', XI IUTAM Congress Paper, Munich, Germany, September 1964.
7. Wang, Chang-Yi, 'Local Instability of Hydrodynamic Cavities', MIT Fluid Dynamics Research Lab., Report No. 63-4, December 1964.
8. Mikhlin, S.G., Integral Equations, Translation by A.H. Armstrong, Macmillan Co., 1964.
9. Song, C.S., 'A Quasi-linear and Linear Theory for Non-Separated and Separated Two-Dimensional, Incompressible, Irrotational Flow About-Lifting Bodies', St. Anthony Falls Lab., T. No.63, Series B, 1963.
10. Cheng, H.K. and Rott, N., 'Generalization of the Inversion Formula of Thin Airfoil Theory', Journal of Rational Mechanics and Analysis, 3, pp. 357-382, 1954.
11. Larock, B.E. and Street, R.L., 'A Non-linear Theory for Fully Cavitating Hydrofoil in a Transverse Gravity Field', Journal of Fluid Mechanics, 29, 2, pp. 317-336, 1967.
12. Agrawal, K.M. and Smith, A.C., 'A Note on Non-linear Models in Two-Dimensional Cavitating Flow', Indian Journal of Pure and Applied Mathematics 3,6, pp. 1256-1268, 1972.
13. Birkhoff, G. and Zarantonello, E.H., Jets, Wakes and Cavities, Academic Press, 1957.

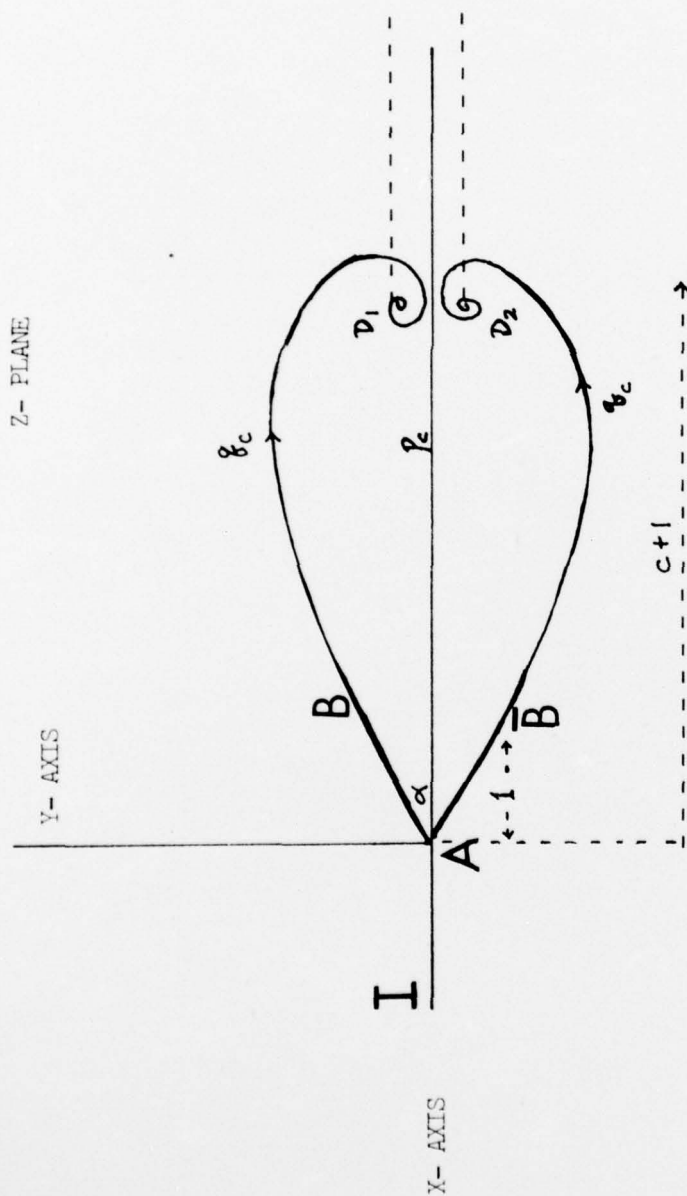


Fig. 1

Physical Flow plane for Fully Cavitating Flow  
(wedge)

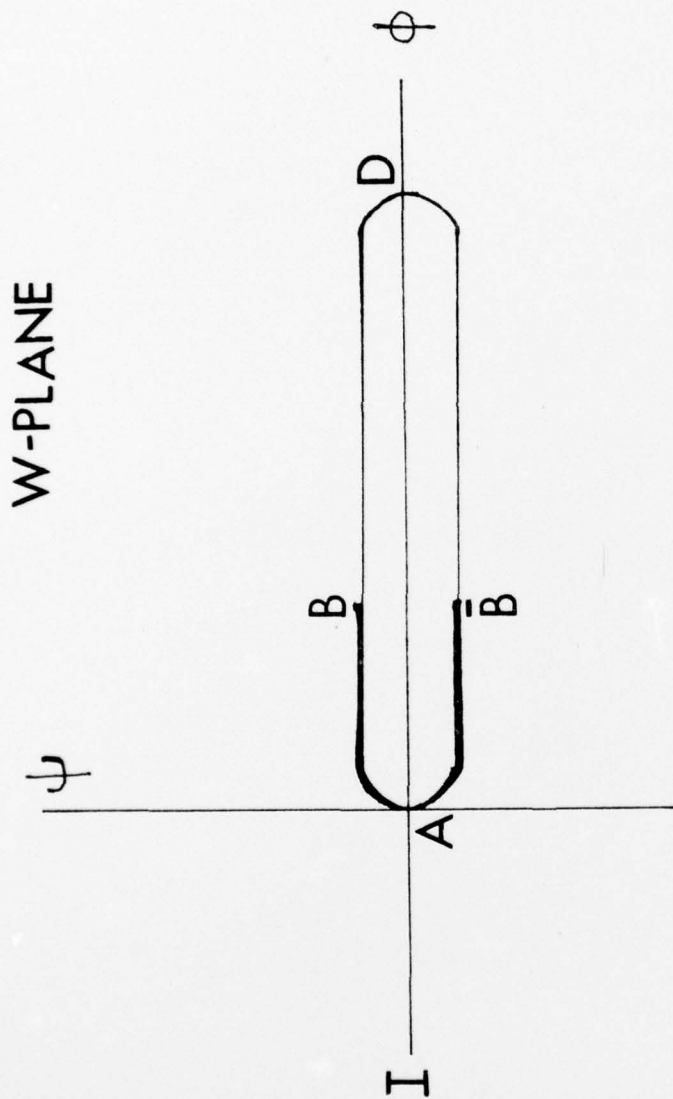


Fig. 2  
Complex potential plane for fully cavitating flow (wedge)

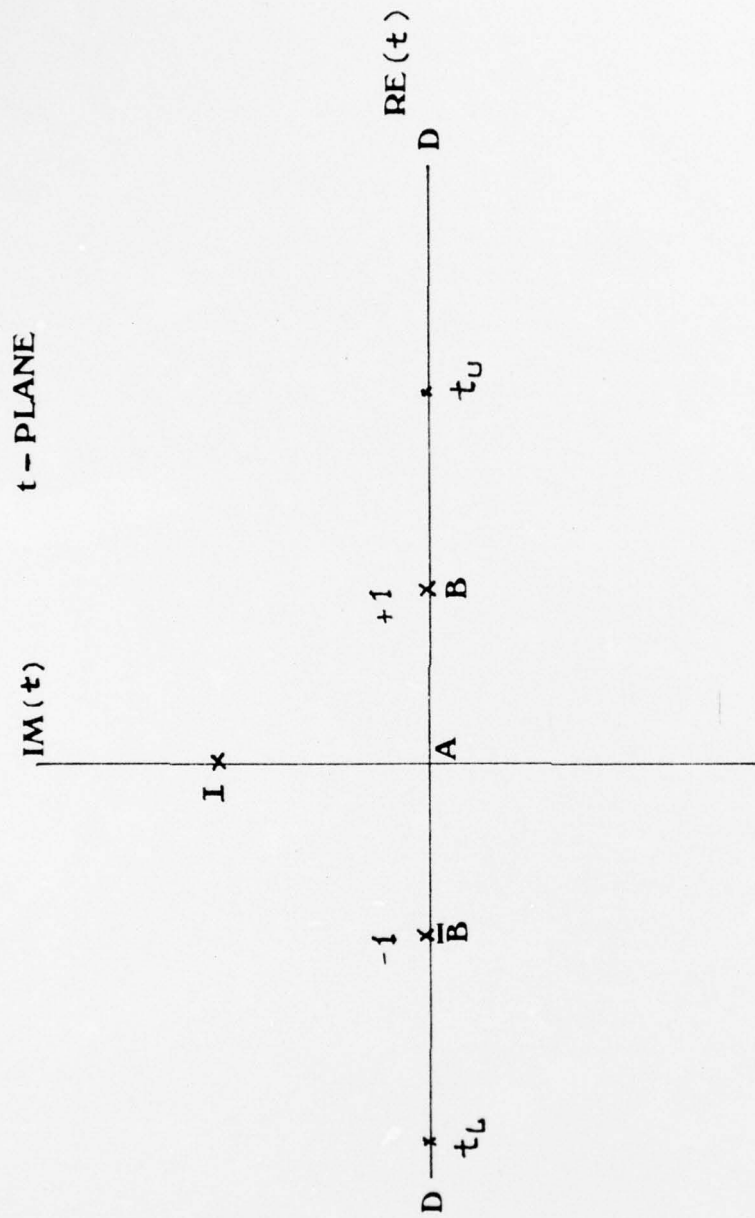


Fig. 3  
Parametric plane for fully cavitating flow (wedge)



No Gravity

$\alpha = 10^\circ$

$C = 22.0$

$C_L = 0$

$C_D = 0.073$

$\sigma = 0.1$

Gravity

$F^2 = 36$

$C = 18.40$

$C_L = -0.13$

$C_D = 0.0711$

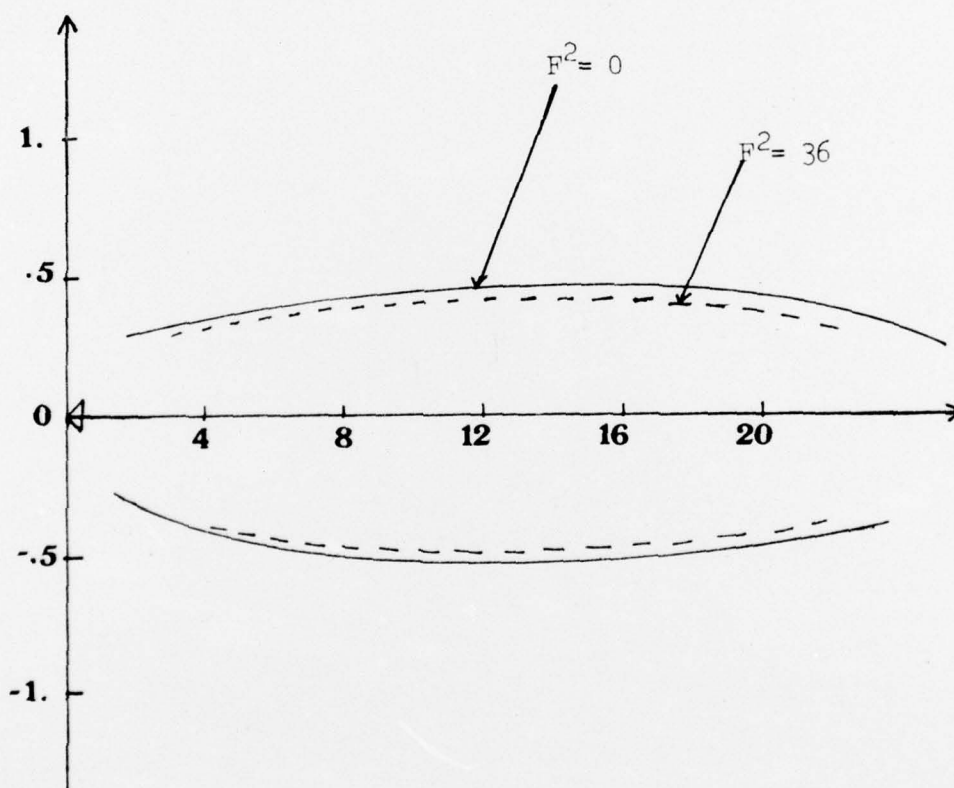


Fig. 4

The effect of Froude number ( $F^2 = 36$ ) on the cavity shape shown by the broken lines.  
 The smooth lines show the cavity shape when ( $F^2 = 0$ )

No Gravity

$\alpha = 5^\circ$

$C = 5.74$

$C_L = 0$

$C_D = 0.0651$

$\sigma = 0.1$

Gravity

$F^2 = 16$

$C = 4.13$

$C_L = -0.015$

$C_D = 0.0645$

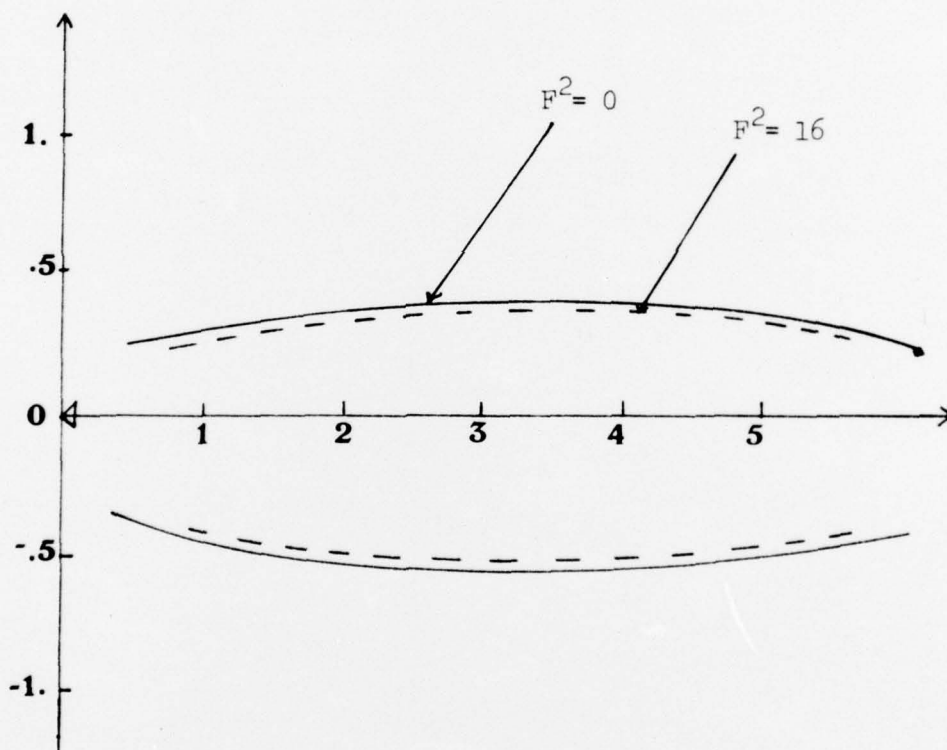


Fig. 5

The effect of Froude number ( $F^2 = 16$ ) on the cavity shape shown by the broken lines.  
 The smooth lines show the cavity shape when  $F^2 = 0$ .

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*egg*